MATH 430, SPRING 2022
NOTES APRIL 18-29

Below we use the following shortcut notation:

- $p|a$ to mean $\phi_{div}(p,a)$; $p \not| a$ to mean $\neg\phi_{div}(p,a)$;
- p_n to mean the n-th prime starting from 0. I.e. the unique x, such that $\phi_{th-prime}(x,n)$ holds. Note that $n \mapsto p_n$ is primitive recursive, and so writing p_n has complexity Δ_1.
- if a codes a sequence of length n, and $i < n$, we use a_i to denote the i-th element of the sequence. I.e. a codes $\langle a_0, ..., a_n \rangle$. In class we showed that saying "$b = a_i$" is equivalent to a Δ_1 formula.

Theorem 1. *There is a Δ_1 formula $\phi_{code-lh}(x,n)$ which says that x codes a sequence of length n.*

Proof. Set $\phi_{code-lh}(x,n) := \phi_{code}(x) \wedge p_{n-1}|x \wedge p_n \not| x$.

□

From now on, if x codes a sequence we use $\mathrm{lh}(n)$ to denote the the length of the sequence. Since $\phi_{code-lh}(x,n)$ is Δ_1, the complexity of $\mathrm{lh}(n)$ is also Δ_1.

Theorem 2. *There is a Δ_1 formula $\phi_{form}(x)$, such that for all $a \in \mathbb{N}$, $\mathfrak{A} \models \phi_{form}[a]$ iff a codes a formula.*

We will skip the proof of this theorem, but recall we had an informal discussion in class why it is true.

Coding notation: for a formula ϕ, the Gödel number of ϕ is the natural number that codes ϕ, denoted by $\ulcorner \phi \urcorner$. Also if $a \in \mathbb{N}$ codes a formula, ϕ_a denotes the formula coded by a. In particular, $\ulcorner \phi_a \urcorner = a$.

Theorem 3. *There is a Δ_1 formula $\phi_{many-form}(x,n)$, such that for all $a \in \mathbb{N}$, $\mathfrak{A} \models \phi_{many-form}[a,n]$ iff a codes a finite sequence of n many formulas. I.e. a codes a sequence $\langle a_0, ..., a_{n-1} \rangle$ and for each $i < n$, a_i codes a formula.*

Proof. Set $\phi_{many-form}(x,n) := \phi_{code}(x) \wedge \mathrm{lh}(x) = n \wedge \forall i < n \phi_{form}(x_i)$.
This is Δ_1, since $\phi_{code}, \mathrm{lh}(x), \phi_{form}, x_i$ are all Δ_1 and and we only used bounded quantifiers. □

Definition 4. *Let T be a set of formulas in the language of PA. T is recursive if $\{e \in \mathbb{N} \mid \phi_e \in T\}$ is a recursive subset of $\mathbb{N}$. We say that T is a recursive extension of PA if $PA \subset T$ and T is recursive.*

Example: one can check that the logical axioms Λ are recursive. Let ϕ_Λ be the Δ_1 formula such that $\Lambda = \{\phi_e \mid \mathfrak{A} \models \phi_\Lambda[e]\}$.

We make one more proposition.

Proposition 5. *There is a Δ_1 formula $\phi_{MP}(x,y,z)$, that says that ϕ_x is the formula $\phi_y \to \phi_z$. More precisely, for all $a,b,c \in \mathbb{N}$, $\mathfrak{A} \models \phi_{MP}[a,b,c]$ iff a,b,c all code formulas and ϕ_a is the formula $\phi_b \to \phi_c$.*

Proof. Set $\phi_{MP}(x,y,z) :=$
$\phi_{form}(x) \wedge \phi_{form}(y) \wedge \phi_{form}(z) \wedge \mathrm{lh}(x) = \mathrm{lh}(y) + 1 + \mathrm{lh}(z) \wedge$
$\forall i < \mathrm{lh}(y)(x_i = y_i) \wedge x_{\mathrm{lh}(y)} = \ulcorner\to\urcorner \wedge \forall i < \mathrm{lh}(z)(x_{\mathrm{lh}(y)+i+1} = z_i)$. [1] □

Theorem 6. *Suppose that T is a recursive extension of PA. Then there is a Δ_1 formula $\phi_{ded-T}(x,y)$, such that for all $e,a \in \mathbb{N}$, $\mathfrak{A} \models \phi_{ded-T}[e,a]$ iff a codes a formula and e codes a deduction from T to ϕ_a.*

Proof. Since T is recursive, let $\phi_T(x)$ be the Δ_1 formula, such that $\{e \in \mathbb{N} \mid \phi_e \in T\} = \{e \in \mathbb{N} \mid \mathfrak{A} \models \phi_T[e]\}$. In other words, $\phi_e \in T$ iff $\mathfrak{A} \models \phi_T[e]$.

Recall that a deduction is a sequence of formulas such that each formula is in $T \cup \Lambda$ or is obtained by modus ponens from earlier formulas in the sequence.

$\phi_{ded-T}(x,y) :=$
$\phi_{form}(y) \wedge \exists n < x(\phi_{many-form}(x,n) \wedge x_{n-1} = y \wedge \forall i < n$
$[\phi_T(x_i) \vee \phi_\Lambda(x_i) \vee \exists j < i \exists k < i(\phi_{MP}(y_j, y_k, y_i))]$
This is Δ_1, since we only used Δ_1 sub-formulas and bounded quantifiers. □

In a similar way, we can define a Δ_1 formula $\phi_{ded-T}(x,y,z)$ to say that y codes a formula with one free variable, say $\psi(v)$ and than x codes a deduction from T to $\psi(z)$. Namely, for all $e,a,b \in \mathbb{N}$,

$$\mathfrak{A} \models \phi_{ded-T}[e,a] \text{ iff } e \text{ codes a deduction from } T \text{ to } \phi_a[b].$$

Definition 7. *Suppose that T is recursive. Set $\phi_{prov-T}(x,y) := \exists e \phi_{ded-T}(e,x,y)$.*

Theorem 8. *If T is a recursive extension of PA, then $\phi_{prov-T}(x,y)$ is a Σ_1 formula, such that*

$$\mathfrak{A} \models \phi_{prov}[a,b] \text{ iff } T \vdash \phi_a[b].$$

Let e be the Gödel number of $\neg\phi_{prov}(x,x)$. In other words, $\phi_e = \neg\phi_{prov}(x,x)$. Define

$$\sigma := \neg\phi_{prov-T}(e,e).$$

Note that σ is exactly $\phi_e(e)$, and informally it says "I am not provable".

Proposition 9. *Suppose that T is a recursive extension of PA*

(1) $\mathfrak{A} \models \sigma$ *iff* $T \not\vdash \sigma$.
(2) *Suppose in addition, that every sentence in T is true in standard arithmetic i.e. $\mathfrak{A} \models T$. Then $\mathfrak{A} \models \sigma$, and so $T \not\vdash \sigma$.*

Proof. The first part is assigned as homework. For the second, suppose for contradiction that $\mathfrak{A} \not\models \sigma$. Then by the first part, we have that $T \vdash \sigma$. Since $\mathfrak{A} \models T$, this means that $\mathfrak{A} \models \sigma$. Contradiction.

□

[1] Here $\ulcorner\to\urcorner$ means the digit corresponding to $\to$ according to some legend fixed in advance.

The sentence σ used above is called the **Gödel sentence for** T. We showed that $\mathfrak{A} \models \sigma$ iff $T \not\vdash \sigma$ iff $\mathfrak{A} \models \neg\phi_{prov-T}(\ulcorner\sigma\urcorner)$.
So, $\mathfrak{A} \models \sigma \leftrightarrow \neg\phi_{prov-T}(\ulcorner\sigma\urcorner)$. We can actually prove something slightly stronger.

Proposition 10. $PA \vdash \sigma \leftrightarrow \neg\phi_{prov-T}(\ulcorner\sigma\urcorner)$

Finally, we can show the first Incompleteness theorem.

Theorem 11. *(Gödel's First Incompleteness Theorem) There is no complete recursive extension T of PA, true in $\mathfrak{A}$. In particular PA is not complete.*

Proof. Fix any recursive extension T of PA, true in $\mathfrak{A}$. By the above proposition, there is a sentence σ true in $\mathfrak{A}$, such that $T \not\vdash \sigma$. T also cannot prove $\neg\sigma$, as $\mathfrak{A} \models \sigma$. It follows that T is incomplete.

□

Now, for the second theorem, define the following formulas. $Incon_T := \phi_{prov-T}(\ulcorner 0 = 1\urcorner)$ and $Con_T := \neg Incon_T$. We will use the following lemma:

Lemma 12. *Let T be as above.*

1. *If $PA \vdash \alpha \to \beta$, then $PA \vdash \phi_{prov-T}(\ulcorner\alpha\urcorner) \to \phi_{prov-T}(\ulcorner\beta\urcorner)$.*
2. *Suppose that ψ is a Σ_1-formula. Then $PA \vdash \psi \to \phi_{prov-T}(\ulcorner\psi\urcorner)$.*
3. *$PA \vdash \phi_{prov-T}(a) \to \phi_{prov-T}(\ulcorner\phi_{prov-T}(a)\urcorner)$.*

Theorem 13. *(Gödel's Second Incompleteness Theorem) Suppose T is a consistent recursive extension of PA. The T does not prove its own consistency.*

Proof. Let σ be the Gödel sentence we used above. First we will show that $PA \vdash Con(T) \to \sigma$. We have:
By Proposition 10, $PA \vdash \phi_{prov-T}(\ulcorner\sigma\urcorner) \to \neg\sigma$;
By Lemma 12 (1) $PA \vdash \phi_{prov-T}(\ulcorner\phi_{prov-T}(\ulcorner\sigma\urcorner)\urcorner) \to \phi_{prov-T}(\ulcorner\neg\sigma\urcorner)$
By Lemma 12 (3), $PA \vdash \phi_{prov-T}(\ulcorner\sigma\urcorner) \to \phi_{prov-T}(\ulcorner\phi_{prov-T}(\ulcorner\sigma\urcorner)\urcorner)$
From all these it follows that :

$$PA \vdash \phi_{prov-T}(\ulcorner\sigma\urcorner) \to \phi_{prov-T}(\ulcorner\neg\sigma\urcorner).$$

Since, trivially, we also have that $PA \vdash \phi_{prov-T}(\ulcorner\sigma\urcorner) \to \phi_{prov-T}(\ulcorner\sigma\urcorner)$, it follows that

$$PA \vdash \phi_{prov-T}(\ulcorner\sigma\urcorner) \to \phi_{prov-T}(\ulcorner 0 = 1\urcorner).$$

In other words, $PA \vdash \neg\sigma \to Incon_T$. Taking the contrapositive, we get $PA \vdash Con_T \to \sigma$.

Suppose now for contradiction $T \vdash Con_T$. Since $PA \subset T$ and $PA \vdash Con_T \to \sigma$, we have that $T \vdash \sigma$. But that contradicts Proposition 9.

□

And so Hilbert's dream that every true mathematical statement can be proved was shattered by Gödel.